

# The Three Most Important Equations in the World (Right Now)

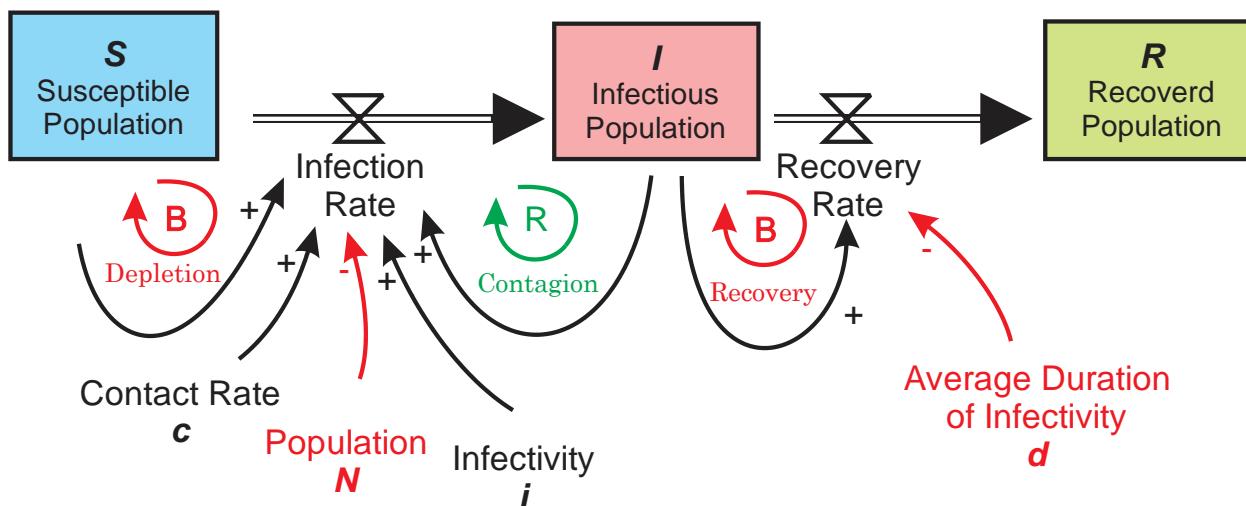
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Public policy on epidemics is now and always has been driven by models of disease spread. There have been many modification and enhancements over the past century, but the benchmark model from which all other depart is the “SIR model”, developed by Kermack and McCormack in 1927. SIR stands for the three “stocks” or “compartments” that constitute the population.  $S$  is the number of susceptible individuals who would come down with a disease if exposed,  $I$  is the number of infectious individuals, and  $R$  is the number of recovered individuals. People “flow” from the stock of susceptible to infectious and from infectious to recovered at variable rates. Those rates are, themselves, a function of the number of people in each stock. (Daley, Daryl. J. and Gani, J. M. 1999)

## The Model

Models such as this are called “system dynamics” models. They simplify the system in question to a set of stocks and flows. The most common way to visualize this as a set of tanks and pipes. As the “tank” of infectious individuals goes up, the rate at which people recover from the disease goes up as well. When flows depend on stocks, the relationship is a “feedback” loop. (Sterman 2018)

Diagrammatically, the SIR model is drawn as follows:



This shows the flow of individuals from susceptible to infectious to recovered. The total population,  $N$ , is fully contained within the three “tanks”, so that  $S + I + R = N$ .

There are three feedback loops. I teach my systems dynamics students to name the feedback loops because it makes it easier to refer back to the different portions of the model. The *depletion* feedback loop controls the stock of susceptibles,  $S$ . This depends on the size of the susceptible population as well as the size of the infectious population. It's a balancing loop for the susceptible population because, as the susceptible population gets smaller the rate of infection gets slower. The *contagion* loop is a reinforcing or positive feedback loop because, as the number of infectious individuals grows, so does the growth rate of infectious individuals. Finally, the *recovery* loop balances the infectious populations because the rate of recovery gets smaller when there are fewer people to recover.

This yields a system of differential equations:

$$\begin{aligned}\frac{dS}{dt} &= -\beta I \frac{S}{N} \\ \frac{dI}{dt} &= \beta I \frac{S}{N} - \gamma I \\ \frac{dR}{dt} &= \gamma R\end{aligned}$$

In words, these equations set the instantaneous change in each population to a value that depends on the populations themselves. Differential equations are characterized by such reflexive relationships. The third, and simplest, says that the rate at which people get better (in terms of people per day) depends only on how many sick people there are. A country or a city with a large number of infectious individuals will have a large number recover every day. (A technicality,  $R$ , can actually be taken to mean *removed*, so that this would include those who die as well.)

The two constants in this epidemic are the recovery rate,  $\gamma$ , and the infectious rate,  $\beta$ . Their inverses are sometimes easier to understand. The inverse of the recovery rate is the *duration* of the infectious period. The infectious rate  $\beta$  is determined by the contact rate and the probability that a contact results in an infection. The *infectious number* for patient zero,  $R_0$ , is given by the ratio of the infectious rate to the recovery rate:  $R_0 = \frac{\beta}{\gamma}$ . If people recover faster than they infect, an epidemic can't take place.

## The Algorithm

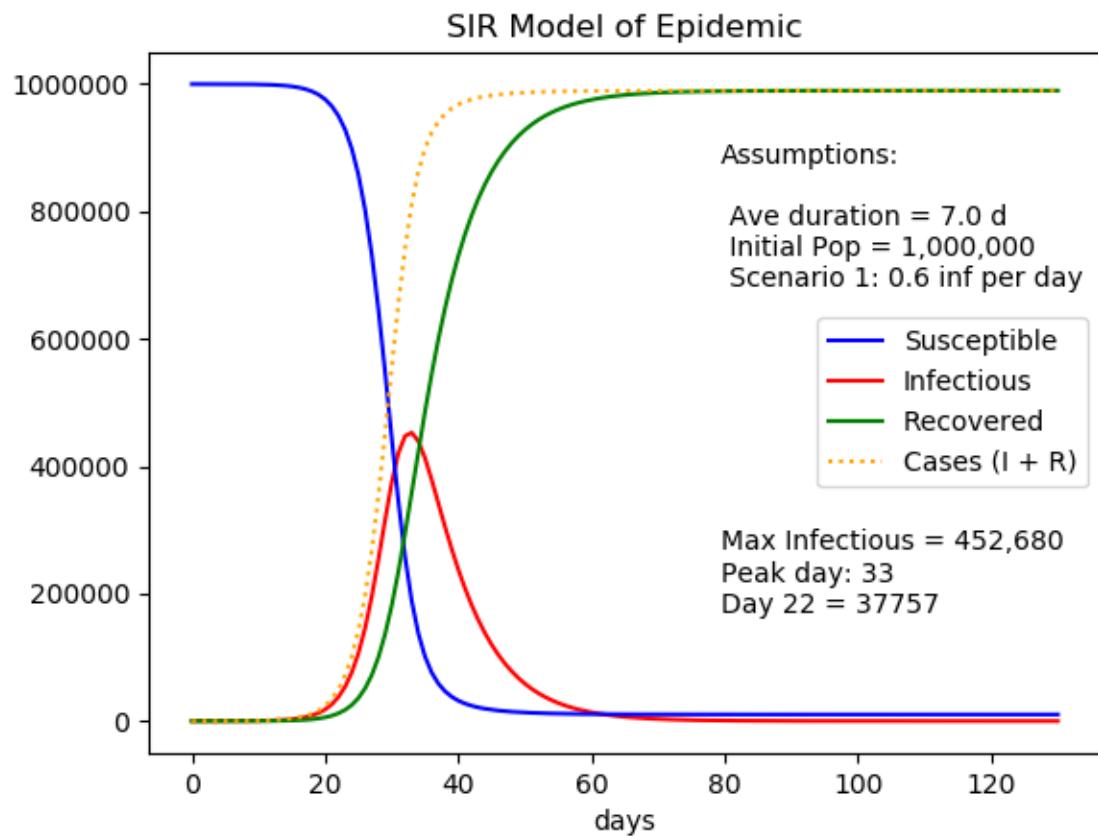
This system of differential equations is called a *second order* system because they include terms that are the product of two variables. The first and second include the product of  $S$  and  $I$ . This means they are difficult to solve. ("Solve" means to express the three stocks as equations in time, depending only on the system constants.) For special cases the SIR equations have been solved, but for the general analysis of the disease it is more typical to *simulate* or *compute* the trajectory. That means, simply, to convert the differential equations to *difference* equations, with the differences set for a unit time step (in this case, days). The SIR difference equations would be:

$$\begin{aligned}\Delta S &= -\beta I \frac{S}{N} \\ \Delta I &= \beta I \frac{S}{N} - \gamma I \\ \Delta R &= \gamma I\end{aligned}$$

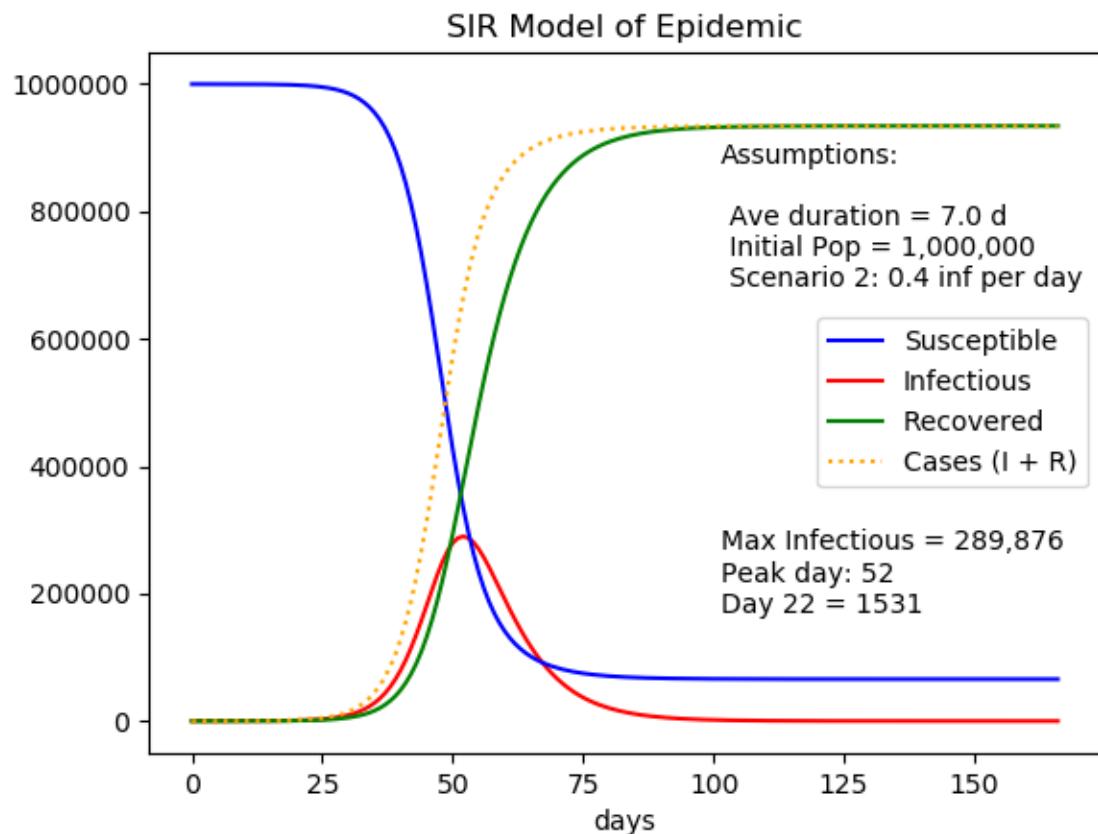
This becomes much simpler to adapt to an algorithm where, for example,  $S_{t+1} = S_t + \Delta S$ .

## The Simulation

The output trajectory of the three stocks in such an algorithm for a seven-day disease and a  $\beta$  of 0.6 infections per day model is given by:



One of the key decision parameters for the model is the reduction of the contact rate. If infections per infectious individual were reduced, the trajectories would look like this:



As a result, one can see that by cutting the contact rate in half, we move from having a peak infectious population of over 450,000 to 290,000, a 64% reduction. We also buy time. The peak day moves from day 33 to day 52.

### Conclusions

The SIR model has been expanded and adjusted extensively. It is still, however, the grandfather of many modern simulations and its key parameter,  $R_0$ , remains one of the modern disease metrics. It has utility in its simplicity. This is demonstrated by the fact that, with merely a few pages of text and three equations, we can demonstrate the primary COVID-19 response strategy of every government. (Chen 2015)

### References

Chen, W. 2015. "A Mathematical Model of Ebola Virus Based on SIR Model." In *2015 International Conference on Industrial Informatics - Computing Technology, Intelligent Technology, Industrial Information Integration (ICIICII)*, 213–16. <https://doi.org/10.1109/ICIICII.2015.135>.

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